



Closedness Type Regularity Conditions in Convex Optimization and Beyond

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The closedness type regularity conditions have proven during the last decade to be viable alternatives to their more restrictive interiority type counterparts, in both convex optimization and different areas where it was successfully applied. In this review article we de- and reconstruct some closedness type regularity conditions formulated by means of epigraphs and subdifferentials, respectively, for general optimization problems in order to stress that they arise naturally when dealing with such problems. The results are then specialized for constrained and unconstrained convex optimization problems. We also hint toward other classes of optimization problems where closedness type regularity conditions were successfully employed and discuss other possible applications of them.

Keywords: convex optimization, duality, closedness type regularity conditions, conjugate functions, epigraphs, subdifferentials

AMS mathematics subject classification. 90C25, 26A51, 90C46.

OPEN ACCESS

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Specialty section:

This article was submitted to
Optimization,
a section of the journal
Frontiers in Applied Mathematics and
Statistics

Received: 25 June 2016

Accepted: 31 August 2016

Published: 16 September 2016

Citation:

Grad S-M (2016) Closedness Type
Regularity Conditions in Convex
Optimization and Beyond.
Front. Appl. Math. Stat. 2:14.
doi: 10.3389/fams.2016.00014

1. INTRODUCTION AND PRELIMINARIES

Regularity conditions play a key role in optimization and nonsmooth analysis, as additional hypotheses that guarantee the fulfillment of strong or converse duality, necessary and/or optimality conditions and various formulae, respectively. In convex optimization there have been proposed a number of regularity conditions (also called constraint qualification when they involve only the constraints of the considered problem), depending on the initial assumptions. For instance, in the differentiable case one employs the regularity conditions due to Abadie, Guignard, or Mangasarian-Fromowitz, while for nondifferentiable constrained optimization problems one has the Slater constraint qualification. However, the latter is a priori violated for many classes of problems where the constraint cone has an empty interior, and several generalizations were proposed for it, the so-called interiority type regularity conditions, that involve notions of generalized interior of a set. But even these fail for large classes of problems and, inspired by Precupanu's pioneering work [1], Burachik, Jeyakumar and their coauthors, and on the other hand Boţ, Wanka and their coauthors have proposed in [2–9] a new class of regularity conditions, the closedness type ones. They have proven first to be sufficient conditions for guaranteeing duality statements in optimization and subdifferential formulae in convex analysis, delivering in the meantime results and formulae in some related research fields as well. They have proven thus to be viable alternatives to their more restrictive interiority type counterparts. In this review paper, that enhances and completes a similar study provided in the book [10], we provide a general look on the usage of the closedness type regularity conditions in the literature until now, showing that they arise naturally while dealing with optimization problems and pointing toward different assertions from the literature that can be rediscovered as special cases of the mentioned general results. To this end we deconstruct and then reconstruct closedness type regularity conditions formulated by means of epigraphs

and subdifferentials, respectively, for general optimization problems, showing afterwards how to particularize them for constrained and unconstrained optimization problems.

As mentioned above, the first papers dealing with closedness type regularity conditions for convex optimization problems were [2–7], having as a starting point earlier statements from Precupanu [1]. Afterwards, it was noticed that other regularity conditions from the literature such as the *Basic Constraint Qualification* (see for instance, [11–13]) or the *Farkas-Minkowski Constraint Qualification* (cf. [14–16]) can be recovered as special cases of some general closedness type regularity conditions.

Let X and Y be locally convex Hausdorff vector spaces, whose topological *dual spaces* are denoted by X^* and Y^* , respectively. The dual spaces can be endowed with different topologies among which the weak* ones, denoted by $\omega(\cdot, \cdot)$ or shortly ω^* , will be considered for formulating closedness type regularity conditions. The natural topology on \mathbb{R} is denoted by \mathcal{R} . By $\langle x^*, x \rangle = x^*(x)$ we denote the value at $x \in X$ of the linear continuous functional $x^* \in X^*$. A *cone* $K \subseteq X$ is a nonempty subset of X which fulfills $\alpha K \subseteq K$ for all $\alpha \geq 0$. A convex cone $K \subseteq X$ induces on X the *partial ordering* “ \leq_K ” defined by $x \leq_K y$ whenever $y - x \in K$, where $x, y \in X$. If $x \leq_K y$ and $x \neq y$ we write $x \leq_K y$. To X a *greatest element* with respect to “ \leq_K ” denoted $\infty_K \notin X$ can be attached and we define $X^\bullet = X \cup \{\infty_K\}$. Then for any $x \in X$ one has $x \leq_K \infty_K$ and we consider on X^\bullet the operations $x + \infty_K = \infty_K + x = \infty_K$ for all $x \in X$ and $\alpha \cdot \infty_K = \infty_K$ for all $\alpha \geq 0$. The *dual cone* of K is $K^* = \{x^* \in X^* : \langle x^*, x \rangle \geq 0 \forall x \in K\}$. By convention, $\langle x^*, \infty_K \rangle = +\infty$ for all $x^* \in K^*$.

Given a subset U of X , by $\text{cl } U$, $\text{int } U$ and $\text{cone } U$ we denote its *closure*, *interior* and *conical hull*, respectively. Moreover, if U is convex, by $\text{sqr } U = \{x \in U : \text{cone}(U - x) \text{ is a closed linear subspace}\}$ we denote its *strong quasi-relative interior*. When $U \subseteq \mathbb{R}^n$ we denote by $\text{ri } U$ its *relative interior*, which coincides in this case with $\text{sqr } U$. The *indicator function* of U is $\delta_U : X \rightarrow \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, defined as $\delta_U(x) = 0$ if $x \in U$ and $\delta_U(x) = +\infty$ otherwise, while its *support function* $\sigma_U : X^* \rightarrow \overline{\mathbb{R}}$ is given by $\sigma_U(x^*) = \sup_{x \in U} \langle x^*, x \rangle$. The *normal cone* associated to the set U at $x \in U$ is $N_U(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0 \forall y \in U\}$ and, for $\varepsilon \geq 0$, the ε -*normal set* of U at $x \in U$ is $N_U^\varepsilon(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \leq -\varepsilon \forall y \in U\}$. The *projection function* of X is $\text{Pr}_X : X \times Y \rightarrow X$, defined by $\text{Pr}_X(x, y) = x$ for $(x, y) \in X \times Y$, the *identity function* of X is $\text{id} : X \rightarrow X$, $\text{id}(x) = x$ for $x \in X$ and, for $n \in \mathbb{N}$, we use the notation $\Delta_{X^n} = \{(x, \dots, x) : x \in X\} \subseteq X^n$.

For being able to deliver by means of duality very general regularity conditions for the problem of guaranteeing the maximal monotonicity of the sum of two maximal monotone operators defined in reflexive Banach spaces, we introduced in Boř et al. [17] the notion of a set closed regarding a subspace (see also [18–20]). For the investigations in this study even more general closedness notions are necessary, that were first proposed in Boncea and Grad [21] (and when $Z = X \times \mathbb{R}$, in [22]; see also [10]).

Definition 1.1. Given $\varepsilon \geq 0$, a set $U \subseteq X \times \mathbb{R}$ is said to be $(0, \varepsilon)$ -*vertically closed regarding the set* $Z \subseteq X \times \mathbb{R}$ if $(\text{cl } U) \cap Z \subseteq (U \cap Z) - (0, \varepsilon)$, while when $Z = X \times \mathbb{R}$, U is called simply $(0, \varepsilon)$ -*vertically closed*. Moreover, a set $U \subseteq X$ that fulfills

$(\text{cl } U) \cap W = U \cap W$, where $W \subseteq X$, is said to be *closed regarding the set* W .

Remark 1. A set $U \subseteq X$ is closed if and only if it is closed regarding the whole space X . Each closed set $U \subseteq X$ is closed regarding any other set $W \subseteq X$, but a set $U \subseteq X$ that is closed regarding some $W \subseteq X$ is not necessarily closed, as shown below in Example 1.

Example 1. The interval $[0, 1] \subseteq \mathbb{R}$ is closed regarding the set $\{0\}$, but not closed.

Example 2. The set $\{0\} \times (0, +\infty) \subseteq \mathbb{R}^2$ is $(0, \varepsilon)$ -vertically closed regarding the set $[-1, 1] \times (-1, +\infty)$ for all $\varepsilon > 0$, but there is no $\varepsilon \geq 0$ for it to be $(0, \varepsilon)$ -vertically closed regarding the set $[-1, 1] \times (-\infty, 0]$. On the other hand, the set $[0, 1] \times (0, +\infty) \subseteq \mathbb{R}^2$ is $(0, \varepsilon)$ -vertically closed for all $\varepsilon > 0$, while the set $[0, 1] \times (0, 1) \subseteq \mathbb{R}^2$ is not $(0, \varepsilon)$ -vertically closed for any $\varepsilon \geq 0$.

Remark 2. In the literature one can find other definitions of ε -closed (see for instance [23, 24]) and *vertically closed* (see [25]) sets, respectively, introduced for purposes that have basically nothing in common with our present investigations.

Let us present now some preliminary notions and results involving functions. Given a function $f : X \rightarrow \overline{\mathbb{R}}$, its *domain* is $\text{dom } f = \{x \in X : f(x) < +\infty\}$, its *epigraph* $\text{epi } f = \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ and *conjugate function* $f^* : X^* \rightarrow \overline{\mathbb{R}}$, $f^*(x^*) = \sup\{\langle x^*, x \rangle - f(x) : x \in X\}$. If $U \subseteq X$, the *conjugate function of f regarding $U \subseteq X$* is $f_U^* : X^* \rightarrow \overline{\mathbb{R}}$, $f_U^* = (f + \delta_U)^*$. We call f *proper* when $f(x) > -\infty$ for all $x \in X$ and $\text{dom } f \neq \emptyset$. When f is proper and $\varepsilon \geq 0$, if $f(x) \in \mathbb{R}$ the $(\text{convex}) \varepsilon$ -*subdifferential* of f at x is $\partial_\varepsilon f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x \rangle - \varepsilon \forall y \in X\}$, while if $f(x) = +\infty$ we take by convention $\partial_\varepsilon f(x) = \emptyset$. The ε -subdifferential of f becomes in case $\varepsilon = 0$ its classical (convex) *subdifferential* denoted by ∂f . When $U \subseteq X$ and $\varepsilon \geq 0$ one has $\partial_\varepsilon \delta_U = N_U^\varepsilon$. The *Young-Fenchel inequality* says that $f_U^*(x^*) + f(x) \geq \langle x^*, x \rangle$ for all $x \in U$ and $x^* \in X^*$. If $U = X$, this inequality is fulfilled as equality if and only if $x^* \in \partial f(x)$ and in general one has $f^*(x^*) + f(x) \leq \langle x^*, x \rangle + \varepsilon$ if and only if $x^* \in \partial_\varepsilon f(x)$. When $f, g : X \rightarrow \overline{\mathbb{R}}$ are proper, their *infimal convolution* is $f \square g : X \rightarrow \overline{\mathbb{R}}$, $f \square g(a) = \inf_{x \in X} [f(x) + g(a - x)]$, that is called *exact* at $a \in X$ when there exists an $x \in X$ such that $f \square g(a) = f(x) + g(a - x)$. For $\alpha \in \mathbb{R}$, defining the function $\alpha f : X \rightarrow \overline{\mathbb{R}}$, $(\alpha f)(x) = \alpha f(x)$ for $x \in X$, we take $0f = \delta_{\text{dom } f}$ when $\alpha = 0$. Given a linear continuous mapping $A : X \rightarrow Y$, its *adjoint* is $A^* : Y^* \rightarrow X^*$, $\langle A^* y^*, x \rangle = \langle y^*, Ax \rangle$ for any $(x, y^*) \in X \times Y^*$, while $\text{Im } A = \{Ax : x \in X\}$ denotes its *image*.

Let $K \subseteq X$ be a convex cone $f : X \rightarrow \overline{\mathbb{R}}$ is said to be K -*increasing* if $f(x) \leq f(y)$ for all $x, y \in X$ such that $x \leq_K y$. A vector function $F : Y \rightarrow X^\bullet$ is said to be *proper* if its *domain* $\text{dom } F = \{y \in Y : F(y) \in X\}$ is nonempty and K -*convex* if $F(tx + (1 - t)y) \leq_K tF(x) + (1 - t)F(y)$ for all $x, y \in Y$ and all $t \in [0, 1]$. When K is closed, it is called K -*epi-closed* if its K -*epigraph* $\text{epi}_K F = \{(y, x) \in Y \times X : x \in F(y) + K\}$ is closed. For $x^* \in K^*$ the function $(x^* F) : Y \rightarrow \overline{\mathbb{R}}$ is defined by $(x^* F)(y) = \langle x^*, F(y) \rangle$, $y \in Y$.

For an attained infimum (supremum) instead of \inf (\sup) we write \min (\max), while the optimal objective value of the optimization problem (P) is denoted by $\nu(P)$.

2. GENERAL PERTURBED SCALAR OPTIMIZATION PROBLEMS

The two main classical duality approaches in convex optimization, the *Lagrange* type for the constrained optimization problems and the *Fenchel* type for the unconstrained ones, have been unified in the general theory of conjugate duality by means of perturbations that can be found, for instance, in Rockafellar [26], Zălinescu [27], Ekeland and Temam [28], and Boţ et al. [29]. Note that also other duality concepts from the literature can be reconsidered in the framework of this general theory, as we have done for the Wolfe and Mond-Weir duality concepts in Boţ et al. [29], Boţ and Grad [30, 31], and Grad and Pop [32] and the geometric duality in Boţ et al. [33]. In the following we present briefly this general duality scheme before proceeding with the ε -duality investigations that lead to constructing closedness type regularity conditions for guaranteeing strong and stable duality, respectively.

Consider two locally convex Hausdorff vector spaces X and Y and the proper function $F : X \rightarrow \overline{\mathbb{R}}$. Let the general optimization problem

$$(PG) \quad \inf_{x \in X} F(x).$$

Using a proper *perturbation function* $\Phi : X \times Y \rightarrow \overline{\mathbb{R}}$, fulfilling $\Phi(x, 0) = F(x)$ for all $x \in X$, a hypothesis that guarantees that $0 \in \text{Pr}_Y \text{ dom } \Phi$, the problem (PG) can be rewritten as

$$(PG) \quad \inf_{x \in X} \Phi(x, 0).$$

We call Y the *perturbation space* and its elements *perturbation variables*. Note that the way Φ is defined guarantees that $0 \in \text{Pr}_Y(\text{dom } \Phi)$. To (PG) one attaches the following *conjugate dual problem* (cf., for instance, [19, 26, 27, 29])

$$(DG) \quad \sup_{y^* \in Y^*} \{-\Phi^*(0, y^*)\},$$

and for this primal-dual pair of optimization problems *weak duality* always holds, i.e., $\nu(DG) \leq \nu(PG)$. In order to investigate further duality properties of these optimization problems, for each $x^* \in X^*$ we consider the following problem

$$(PG_{x^*}) \quad \inf_{x \in X} [\Phi(x, 0) - \langle x^*, x \rangle],$$

obtained by linearly perturbing the objective function of (PG). Thus (PG) is embedded in the family of optimization problems $\{(PG_{x^*}) : x^* \in X^*\}$, where it coincides with (PG_0) . To each problem in the mentioned family one can attach the corresponding conjugate dual problem, namely, for $x^* \in X^*$,

$$(DG_{x^*}) \quad \sup_{y^* \in Y^*} \{-\Phi^*(x^*, y^*)\}.$$

By construction, whenever $x^* \in X^*$ one has $\nu(DG_{x^*}) \leq \nu(PG_{x^*})$, but more important is to find out when the optimal objective values of the primal and its corresponding dual problem coincide or they are different with less than a given small $\varepsilon \geq 0$.

Definition 2.1. Let $\varepsilon \geq 0$. We say that there is ε -*duality gap* for the problems (PG) and (DG) if $\nu(PG) - \nu(DG) \leq \varepsilon$. If $\nu(PG_{x^*}) - \nu(DG_{x^*}) \leq \varepsilon$ for all $x^* \in X^*$, we say that for (PG) and (DG) one has *stable ε -duality gap*.

Definition 2.2. We say that there is *strong duality* for the problems (PG) and (DG) if $\nu(PG) = \nu(DG)$ and (DG) has an optimal solution. When there is strong duality for (PG) and (DG) and (PG) has an optimal solution, too, one speaks about *total duality*. If $\nu(PG_{x^*}) = \nu(DG_{x^*})$ and (DG_{x^*}) has an optimal solution for all $x^* \in X^*$, we say that for (PG) and (DG) one has *stable strong duality*.

Definition 2.3. Let $\varepsilon \geq 0$. An element $x \in X$ is said to be an ε -*optimal solution* to (PG) if $0 \in \partial_\varepsilon \Phi(\cdot, 0)(x)$.

In order to ensure strong duality for (PG) and (DG) one usually assumes that Φ is convex and a certain regularity condition is fulfilled. Various such additional hypotheses were considered in the literature (some of which are mentioned later in Remark 8), the most relevant being the interiority type ones (mentioned in [19, 27]) and the closedness type ones (cf. [3, 4, 6, 7, 10, 19, 29]). Moreover, the situation of total duality is closely related to some subdifferential formulae and the regularity conditions can be used to guarantee these, too. However, in some situations one can only show that the difference between the optimal objective values of the primal and dual problem is less than an $\varepsilon \geq 0$, situation coined in Boncea and Grad [21, 22] as ε -duality gap. Using as a basis our investigations in these articles, where the ε -duality gap for composed and constrained optimization problems, respectively, were characterized via epigraph and subdifferential inclusions, we provide in the following section characterizations via epigraph inclusions of stable ε -duality gap (for an $\varepsilon \geq 0$) for very general optimization problems, with the involved functions not necessary convex, only proper. Endowing then the functions with the classical convexity and topological properties, we derive new important equivalences as well as sufficient conditions, from which when $\varepsilon = 0$ *closedness type regularity conditions* are derived. These can then be employed, for instance, for subdifferential formulae, as done in Boţ et al. [18], Hiriart-Urruty [34], and Boţ and Grad [35] or, like in Boţ and Grad [36] and Grad and Wanka [37], for providing formulae for biconjugates of combinations of functions.

After presenting these investigations for general optimization problems, we deal with both *constrained* and *unconstrained optimization problems*, showing how the mentioned results can be specialized for them, too, by means of the perturbation theory (cf. [26, 27]). In this way some of our results from Grad [10], Boţ et al. [18, 20, 29, 38, 39], Boncea and Grad [21, 22], and Boţ and Grad [35] as well as different others from the literature (e.g., from [2–4, 6, 7, 19]) can be obtained as special cases of the general statements presented below.

Before proceeding, we recall a statement that will play an important role later in our investigations when additional convexity and topological hypotheses will be considered on the function Φ in order to derive closedness type regularity conditions for various duality situations.

Lemma 2.1. (cf. [40, Theorem 2.2 and Theorem 2.3], see also [19, Theorem 5.1 and Theorem 5.2] and [1]) Let the function Φ be also convex and lower semicontinuous. Then one has $(\Phi(\cdot, 0))^* = \text{cl}_{\omega^*} \inf_{y^* \in Y^*} \Phi^*(\cdot, y^*)$ and

$$\text{epi}((\Phi(\cdot, 0))^*) = \text{cl}_{\omega^*} \bigcup_{y^* \in Y^*} \text{epi}(\Phi^*(\cdot, y^*)) = \text{cl}_{\omega^*} \text{Pr}_{X^* \times \mathbb{R}} \text{epi } \Phi^*.$$

To make this review paper as self-contained as possible, we recall the definition of the Lagrangian function for the pair of primal-dual problems $(PG_{x^*}) - (DG_{x^*})$, where $x^* \in X^*$, and a nice connection between it and the optimal objective values of the mentioned problems (cf. [26, 27, 29]).

Definition 2.4. Let $x^* \in X^*$. The function $L^{(PG_{x^*})} : X \times Y^* \rightarrow \overline{\mathbb{R}}$ defined by

$$L^{(PG_{x^*})}(x, y^*) = \inf_{y \in Y} [\Phi(x, y) - \langle x^*, x \rangle - \langle y^*, y \rangle]$$

is called the *Lagrangian function* of the pair of primal-dual problems $(PG_{x^*}) - (DG_{x^*})$ relative to the perturbation function Φ .

Remark 3. Given $x^* \in X^*$, one can rewrite the primal-dual pair of problems $(PG_{x^*}) - (DG_{x^*})$ by means of the Lagrangian $L^{(PG_{x^*})}$ as follows. The dual problem (DG_{x^*}) is equivalent to $\sup_{y^* \in Y^*} \inf_{x \in X} L^{(PG_{x^*})}$, while if for any $x \in X$ the function $\Phi(x, \cdot)$ is convex, lower semicontinuous and nowhere equal to $-\infty$, (PG_{x^*}) actually means $\inf_{x \in X} \sup_{y^* \in Y^*} L^{(PG_{x^*})}$. Note that even without the additional hypotheses, $v(PG_{x^*}) \geq \inf_{x \in X} \sup_{y^* \in Y^*} L^{(PG_{x^*})}$.

3. CHARACTERIZATIONS INVOLVING EPIGRAPHS

Most of the closedness type regularity conditions that can be found in the literature are constructed by means of epigraphs and this section is dedicated to them.

Let $\varepsilon \geq 0$. We begin our presentation with a characterization via epigraph inclusions of a situation of partial stable ε -duality gap for the problems (PG) and (DG) that holds for very general proper functions, no convexity or topological assumptions being necessary in order to derive it. Despite its simple formulation and proof, this statement is a key result for the following presentation and one shall see that using it virtually all the statements involving closedness type regularity conditions from the literature on duality for convex optimization problems, some of which have quite involved proofs in the original sources, can be rediscovered.

Theorem 3.1. Given a subset W of X^* , one has

$$\text{epi}(\Phi(\cdot, 0))^* \cap (W \times \mathbb{R}) \subseteq \text{Pr}_{X^* \times \mathbb{R}} \text{epi } \Phi^* \cap (W \times \mathbb{R}) - (0, \varepsilon) \quad (1)$$

if and only if for each $x^* \in W$ there exists a $\bar{y}^* \in Y^*$ such that

$$(\Phi(\cdot, 0))^*(x^*) \geq \Phi^*(x^*, \bar{y}^*) - \varepsilon. \quad (2)$$

PROOF. If W is empty or for $x^* \in W$ it holds $(\Phi(\cdot, 0))^*(x^*) = +\infty$, there is nothing to prove, since subtracting anything from an empty set gives again the empty set and $(\Phi(\cdot, 0))^* \leq \inf_{y^* \in Y^*} \Phi^*(\cdot, y^*)$, respectively.

Let $x^* \in W$ such that $(\Phi(\cdot, 0))^*(x^*) \in \mathbb{R}$. Then $(x^*, (\Phi(\cdot, 0))^*(x^*)) \in \text{epi}(\Phi(\cdot, 0))^*$. For $\bar{y}^* \in Y^*$ one has $(x^*, \bar{y}^*, (\Phi(\cdot, 0))^*(x^*)) \in \text{epi } \Phi^* - (0, 0, \varepsilon)$ if and only if $\Phi^*(x^*, \bar{y}^*) \leq (\Phi(\cdot, 0))^*(x^*) + \varepsilon$, hence the desired equivalence. \square

Remark 4. One can rewrite the inequality (Equation 2) as $-(\Phi(\cdot, 0))^*(x^*) \leq -\Phi^*(x^*, \bar{y}^*) + \varepsilon$, where in the left-hand side there is actually $v(PG_{x^*})$. However, in the right-hand side one does not necessarily have $v(DG_{x^*}) + \varepsilon$, since there is no guarantee that the supremum in (DG_{x^*}) is attained at \bar{y}^* . Consequently, Equation (2) implies $v(PG_{x^*}) \leq v(DG_{x^*}) + \varepsilon$, i.e., condition (1) implies that there is a partial (because $x^* \in W \subseteq X^*$) stable ε -duality gap for the problems (PG) and (DG) . Note also that employing (Equation 2), one can characterize the stable ε -duality gap for the primal-dual pair $(PG) - (DG)$ via the epigraph inclusion

$$\text{epi}(\Phi(\cdot, 0))^* \subseteq \text{epi } \inf_{y^* \in Y^*} \Phi^*(\cdot, y^*) - (0, \varepsilon). \quad (3)$$

When $\varepsilon = 0$ the epigraph inclusion and the inequality from Theorem 3.1 collapse both into equalities.

Corollary 3.2. Given a subset W of X^* , one has

$$\text{epi}(\Phi(\cdot, 0))^* \cap (W \times \mathbb{R}) = \text{Pr}_{X^* \times \mathbb{R}} \text{epi } \Phi^* \cap (W \times \mathbb{R}) \quad (4)$$

if and only if for each $x^* \in W$ there exists a $\bar{y}^* \in Y^*$ such that

$$(\Phi(\cdot, 0))^*(x^*) = \Phi^*(x^*, \bar{y}^*).$$

Taking $W = X^*$, one obtains an epigraph inclusion that ensures the stable ε -duality gap for (PG) and (DG) , that (taking into consideration Corollary 3.2) becomes stable strong duality when $\varepsilon = 0$.

Corollary 3.3. It holds

$$\text{epi}(\Phi(\cdot, 0))^* \subseteq \text{Pr}_{X^* \times \mathbb{R}} \text{epi } \Phi^* - (0, \varepsilon) \quad (5)$$

if and only if for each $x^* \in X^*$ there exists a $\bar{y}^* \in Y^*$ such that

$$(\Phi(\cdot, 0))^*(x^*) \geq \Phi^*(x^*, \bar{y}^*) - \varepsilon.$$

Corollary 3.4. *It holds*

$$\text{epi}(\Phi(\cdot, 0))^* = \text{Pr}_{X^* \times \mathbb{R}} \text{epi } \Phi^*$$

if and only if for each $x^* \in X^*$ there exists a $\bar{y}^* \in Y^*$ such that

$$(\Phi(\cdot, 0))^*(x^*) = \Phi^*(x^*, \bar{y}^*) = \min_{y^* \in Y^*} \Phi^*(x^*, y^*).$$

Using Theorem 3.1, one can derive necessary and sufficient ε -optimality conditions for primal-dual pairs $(PG_{x^*}) - (DG_{x^*})$, where $x^* \in X^*$.

Theorem 3.5. *Let W be a subset of X^* and $x^* \in W$.*

(a) *If $\bar{x} \in X$ is an optimal solution to (PG_{x^*}) and Equation (1) is satisfied, then there exists an ε -optimal solution $\bar{y}^* \in Y^*$ to (DG_{x^*}) , such that*

$$\Phi(\bar{x}, 0) + \Phi^*(x^*, \bar{y}^*) \leq \langle x^*, \bar{x} \rangle + \varepsilon, \tag{6}$$

or, equivalently,

$$(x^*, \bar{y}^*) \in \partial_\varepsilon \Phi(\bar{x}, 0). \tag{7}$$

(b) *Assume that $\bar{x} \in X$ and $\bar{y}^* \in Y^*$ fulfill (Equations 6 or 7). Then \bar{x} is an ε -optimal solution to (PG_{x^*}) , \bar{y}^* is an ε -optimal solution to (DG_{x^*}) and $v(PG_{x^*}) \leq v(DG_{x^*}) + \varepsilon$.*

PROOF. (a) From Theorem 3.1 and Remark 4 one obtains $\Phi(\bar{x}, 0) + \Phi^*(x^*, \bar{y}^*) \leq \langle x^*, \bar{x} \rangle + \varepsilon$. The weak duality for (PG_{x^*}) and (DG_{x^*}) yields $v(DG_{x^*}) \leq -\Phi^*(x^*, \bar{y}^*) + \varepsilon$, i.e., \bar{y}^* is an ε -optimal solution to (DG_{x^*}) .

(b) From Equation (6) one gets $\Phi(\bar{x}, 0) - \langle x^*, \bar{x} \rangle \leq \varepsilon - \Phi^*(0, \bar{y}^*)$, that, employing also the weak duality, implies $v(DG_{x^*}) \leq v(PG_{x^*}) \leq \Phi(\bar{x}, 0) - \langle x^*, \bar{x} \rangle \leq \varepsilon - \Phi^*(0, \bar{y}^*) \leq v(DG_{x^*}) + \varepsilon \leq v(PG_{x^*}) + \varepsilon$. Then \bar{x} is an ε -optimal solution to (PG_{x^*}) and \bar{y}^* one to (DG_{x^*}) . □

Remark 5. Note that in Theorem 3.5 one does not obtain the kind of equivalence usually delivered in optimality conditions statements as in a) one assumes \bar{x} to be an optimal solution to (PG_{x^*}) , while in b) \bar{x} turns out to be only an ε -optimal solution to (PG_{x^*}) .

When $\varepsilon = 0$, relations (Equations 6 and 7) become *optimality conditions* for (PG_{x^*}) and (DG_{x^*}) .

Corollary 3.6. *Let W be a subset of X^* and $x^* \in W$.*

(a) *If $\bar{x} \in X$ is an optimal solution to (PG_{x^*}) and the condition (4) is satisfied, then there exists an optimal solution $\bar{y}^* \in Y^*$ to (DG_{x^*}) , such that*

$$\Phi(\bar{x}, 0) + \Phi^*(x^*, \bar{y}^*) = \langle x^*, \bar{x} \rangle, \tag{8}$$

or, equivalently,

$$(x^*, \bar{y}^*) \in \partial \Phi(\bar{x}, 0). \tag{9}$$

(b) *Assume that $\bar{x} \in X$ and $\bar{y}^* \in Y^*$ fulfill (Equations 8 or 9). Then \bar{x} is an optimal solution to (PG_{x^*}) , \bar{y}^* is an optimal solution to (DG_{x^*}) and $v(PG_{x^*}) = v(DG_{x^*})$.*

Remark 6. When $W = X^*$, Corollary 3.6 delivers *stable optimality conditions* for (PG) and (DG) .

Remark 7. Taking $x^* = 0$ in Theorem 3.5 one obtains ε -optimality conditions for the primal-dual pair of optimization problems $(PG) - (DG)$ and, moreover, (or, alternatively, directly from Corollary 3.3) that the satisfaction of the condition (5) guarantees that there is ε -duality gap for these problems.

So far we have *deconstructed* the closedness type regularity conditions formulated by means of epigraph inclusions, showing that such inclusions are intimately connected to ε -duality gap statements. In the following we will *reconstruct* them and for this we add convexity and topological properties to the function Φ .

We begin with a characterization of Equation (2) by means of the notion of the $(0, \varepsilon)$ -vertical closedness of the conjugate of Φ regarding a product set that can be obtained via Lemma 2.1 and Theorem 3.1.

Theorem 3.7. *Let W be a subset of X^* and take the function Φ also convex and lower semicontinuous. Then the set $\text{Pr}_{X^* \times \mathbb{R}} \text{epi } \Phi^*$ is $(0, \varepsilon)$ -vertically closed regarding $W \times \mathbb{R}$ in the topology $\omega(X^*, X) \times \mathcal{R}$ if and only if for every $x^* \in W$ there exists a $\bar{y}^* \in Y^*$ such that Equation (2) holds.*

When $\varepsilon = 0$ Theorem 3.7 collapses to Boř [19, Theorem 9.1].

Corollary 3.8. *Let W be a subset of X^* and the function Φ be also convex and lower semicontinuous. Then the set $\text{Pr}_{X^* \times \mathbb{R}} \text{epi } \Phi^*$ is closed regarding the set $W \times \mathbb{R}$ in the topology $\omega(X^*, X) \times \mathcal{R}$ if and only if for each $x^* \in W$ there exists a $\bar{y}^* \in Y^*$ such that*

$$(\Phi(\cdot, 0))^*(x^*) = \Phi^*(x^*, \bar{y}^*) = \min_{y^* \in Y^*} \Phi^*(x^*, y^*).$$

Taking in Theorem 3.7 $W = X^*$ one gets the following statement.

Corollary 3.9. *Let the function Φ be also convex and lower semicontinuous. Then the set $\text{Pr}_{X^* \times \mathbb{R}} \text{epi } \Phi^*$ is $(0, \varepsilon)$ -vertically closed in the topology $\omega(X^*, X) \times \mathcal{R}$ if and only if for each $x^* \in X^*$ there exists a $\bar{y}^* \in Y^*$ such that*

$$(\Phi(\cdot, 0))^*(x^*) \geq \Phi^*(x^*, \bar{y}^*) - \varepsilon.$$

Taking in Corollary 3.8 moreover $W = X^*$ (or $\varepsilon = 0$ in Corollary 3.9, noticing also the comment preceding Corollary 3.2), one obtains a characterization of the stable strong duality for (PG) and (DG) , rediscovering thus [29, Theorem 3.2.2] and [4, Theorem 3.1].

Corollary 3.10. *Let the function Φ be also convex and lower semicontinuous. Then the set $\text{Pr}_{X^* \times \mathbb{R}} \text{epi } \Phi^*$ is closed in the topology $\omega(X^*, X) \times \mathcal{R}$ if and only if for each $x^* \in X^*$ there exists a $\bar{y}^* \in Y^*$ such that*

$$(\Phi(\cdot, 0))^*(x^*) = \Phi^*(x^*, \bar{y}^*) = \min_{y^* \in Y^*} \Phi^*(x^*, y^*).$$

A crucial consequence of Theorem 3.7, via Corollary 3.10, is the strong duality statement for (PG) and (DG) that follows (see also [1, 3, 4, 19, 29]) and contains the weakest hypotheses that guarantee this outcome.

Corollary 3.11. Assume that Φ is convex and lower semicontinuous. If $\text{Pr}_{X^* \times \mathbb{R}} \text{epi } \Phi^*$ is a closed set in the topology $\omega(X^*, X) \times \mathcal{R}$, then $v(PG) = v(DG)$ and the dual problem (DG) has an optimal solution $\bar{y}^* \in Y^*$.

Remark 8. Several regularity conditions were proposed in the literature in order to achieve strong duality for (PG) and (DG). We list in the following the most important of those considered when the function Φ is convex (cf. [19, 27, 29]), namely the one involving continuity

$$(RC_1^G) \quad \left| \begin{array}{l} \exists x' \in X \text{ such that } (x', 0) \in \text{dom } \Phi \text{ and } \Phi(x', \cdot) \\ \text{is continuous at } 0, \end{array} \right.$$

a weak generalized interiority type one,

$$(RC_2^G) \quad \left| \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces, } \Phi \text{ is lower} \\ \text{semicontinuous and } 0 \in \text{sqr } \text{Pr}_Y(\text{dom } \Phi), \end{array} \right.$$

another one applicable when the dimension of the linear hull of $\text{Pr}_Y(\text{dom } \Phi)$ is finite,

$$(RC_3^G) \quad \left| \quad 0 \in \text{ri } \text{Pr}_Y(\text{dom } \Phi), \right.$$

and finally the closedness type regularity condition already mentioned in Corollary 3.11,

$$(RC_4^G) \quad \left| \begin{array}{l} \Phi \text{ is lower semicontinuous and } \text{Pr}_{X^* \times \mathbb{R}}(\text{epi } \Phi^*) \text{ is} \\ \text{closed in the topology } \omega(X^*, X) \times \mathcal{R}. \end{array} \right.$$

Worth noticing is that all these four regularity conditions ensure actually stable strong duality for the primal-dual pair of optimization problems (PG) – (DG). One can thus notice that (RC_i^G) , $i = 1, 2, 3$, imply (RC_4^G) , which is equivalent to the stable strong duality for (PG) and (DG). An example to show that the closedness type regularity condition (RC_4^G) is indeed weaker than its counterparts of continuity or interiority type follows, others being available for instance in Boţ et al. [18, 20].

Example 3. (cf. [6]) Let $X = Y = \mathbb{R}$ and $\Phi : \mathbb{R} \times \mathbb{R} \rightarrow \bar{\mathbb{R}}$, $\Phi(x, y) = \delta_{\mathbb{R}_+}(x) + \delta_{\mathbb{R}_-}(x+y)$. Then $\text{Pr}_{X^* \times \mathbb{R}} \text{epi } \Phi^* = \mathbb{R} \times \mathbb{R}_+$ is closed, thus (RC_4^G) is satisfied, while neither is $\Phi(0, \cdot)$ continuous at 0 nor is $0 \in \text{ri } \text{Pr}_Y(\text{dom } \Phi) = (-\infty, 0)$ fulfilled, hence (RC_i^G) , $i = 1, 2, 3$, fail in this case.

Necessary and sufficient optimality conditions for (PG_{x^*}) and (DG_{x^*}) , where $x^* \in W \subseteq X^*$ follow by means of Theorem 3.5 via Corollary 3.10 (see also [19, 27, 29]).

Corollary 3.12. Let W be a subset of X^* and $x^* \in W$.

(a) Assume that Φ is convex. Let $\bar{x} \in X$ be an optimal solution to (PG_{x^*}) and assume that one of the regularity conditions (RC_i^G) , $i \in \{1, 2, 3, 4\}$, is fulfilled. Then there exists a $\bar{y}^* \in Y^*$, an optimal solution to (DG_{x^*}) , such that one has

$$\Phi(\bar{x}, 0) + \Phi^*(x^*, \bar{y}^*) = \langle x^*, \bar{x} \rangle, \quad (10)$$

or, equivalently,

$$(x^*, \bar{y}^*) \in \partial \Phi(\bar{x}, 0). \quad (11)$$

(b) Assume that $\bar{x} \in X$ and $\bar{y}^* \in Y^*$ fulfill (Equation 10) or (Equation 11). Then \bar{x} is an optimal solution to (PG_{x^*}) , \bar{y}^* is an optimal solution to (DG_{x^*}) and $v(PG_{x^*}) = v(DG_{x^*})$.

Remark 9. When $W = X^*$, Corollary 3.12 delivers what may be called stable optimality conditions for (PG) and (DG). Taking there $x^* = 0$ one obtains necessary and sufficient optimality conditions for (PG) and (DG) (see also [19, 27, 29]).

As byproducts of the duality investigations presented in this subsection one can also derive ε -Farkas statements and results involving (η, ε) -saddle points, inspired for instance by Boţ and Wanka [41], as follows. We begin with the ε -Farkas type results for (PG_{x^*}) and (DG_{x^*}) , where $x^* \in W \subseteq X^*$. They extend some recent Farkas type statements from the literature that generalize the classical Farkas Lemma. For more on the latest developments in the literature on Farkas type statements we refer the reader to the survey [42] together with the references therein and the additional discussion from the same issue of the journal.

Theorem 3.13. Let W be a subset of X^* .

- (a) Suppose the validity of Equation (1). For $x^* \in W$, if one has $\Phi(x, 0) - \langle x^*, x \rangle \geq \varepsilon/2$ for all $x \in X$, then there exists a $\bar{y}^* \in Y^*$ such that $\Phi^*(x^*, \bar{y}^*) \leq \varepsilon/2$.
- (b) For $x^* \in W$, if there exists a $\bar{y}^* \in Y^*$ such that $\Phi^*(x^*, \bar{y}^*) \leq -\varepsilon/2$, then $\Phi(x, 0) - \langle x^*, x \rangle \geq \varepsilon/2$ for all $x \in X$.

PROOF. (a) The existence of $\bar{y}^* \in Y^*$ such that $-(\Phi(\cdot, 0))^*(x^*) \leq \varepsilon - \Phi^*(x^*, \bar{y}^*)$ is guaranteed by Theorem 3.1. Then $\varepsilon/2 \leq \varepsilon - \Phi^*(x^*, \bar{y}^*)$ and the conclusion follows.

(b) The weak duality for (PG_{x^*}) and (DG_{x^*}) yields $\Phi(x, 0) - \langle x^*, x \rangle \geq -\Phi^*(x^*, \bar{y}^*) \geq \varepsilon/2$. □

Using Equation (3) as a regularity condition other ε -Farkas type results for (PG_{x^*}) and (DG_{x^*}) , where $x^* \in W \subseteq X^*$, can be formulated and proven analogously to the ones in Theorem 3.13.

Theorem 3.14. Let W be a subset of X^* .

- (a) Suppose that Equation (3) holds. For $x^* \in W$, if $\Phi(x, 0) - \langle x^*, x \rangle \geq \varepsilon/2$ for all $x \in X$ then $\inf_{y^* \in Y^*} \Phi^*(x^*, y^*) \leq \varepsilon/2$.
- (b) Given $x^* \in W$, if $\inf_{y^* \in Y^*} \Phi^*(x^*, y^*) \leq -\varepsilon/2$, then $\Phi(x, 0) - \langle x^*, x \rangle \geq \varepsilon/2$ for all $x \in X$.

If $\varepsilon = 0$, the ε -Farkas type results become equivalences, as follows.

Corollary 3.15. Let W be a subset of X^* and suppose that Equation (4) holds. Given $x^* \in W$, one has $\Phi(x, 0) - \langle x^*, x \rangle \geq 0$ for all $x \in X$ if and only if there exists a $\bar{y}^* \in Y^*$ such that $\Phi^*(x^*, \bar{y}^*) \leq 0$.

In the following statement we assume that Equation (3) holds as an equality for $\varepsilon = 0$.

Corollary 3.16. *Let W be a subset of X^* and suppose that $\text{epi}(\Phi(\cdot, 0))^* = \text{epi} \inf_{y^* \in Y^*} \Phi^*(\cdot, y^*)$. Given $x^* \in W$, one has $\Phi(x, 0) - \langle x^*, x \rangle \geq 0$ for all $x \in X$ if and only if $\inf_{y^* \in Y^*} \Phi^*(x^*, y^*) \leq 0$.*

In order to deal with statements involving (η, ε) -saddle points, we generalize the classical notion of a saddle point (cf. [10, 21]).

Definition 3.1. Let $\eta \geq 0$ and $x^* \in X^*$. We say that $(\bar{x}, \bar{y}^*) \in X \times Y^*$ is an (η, ε) -saddle point of the Lagrangian $L^{(PG_{x^*})}$ if

$$L^{(PG_{x^*})}(\bar{x}, y^*) - \eta \leq L^{(PG_{x^*})}(\bar{x}, \bar{y}^*) \leq L^{(PG_{x^*})}(x, \bar{y}^*) + \varepsilon \forall (x, y^*) \in X \times Y^*.$$

Remark 10. The notion of an ε -saddle point of a function with two variables was already considered in the literature, see for instance [43, 44].

Slightly weakening the properness hypothesis imposed on Φ and adding to it convexity and topological assumptions, one obtains the following statement connecting the (η, ε) -saddle points of $L^{(PG_{x^*})}$ with the $(\varepsilon + \eta)$ -duality gap for the problems (PG_{x^*}) and (DG_{x^*}) , and the existence of some $(\varepsilon + \eta)$ -optimal solutions to them.

Theorem 3.17. *Let $\eta \geq 0$ and $x^* \in X^*$.*

- (a) *If $(\bar{x}, \bar{y}^*) \in X \times Y^*$ is an (η, ε) -saddle point of $L^{(PG_{x^*})}$ and $\Phi(\bar{x}, \cdot)$ is convex, lower semicontinuous and nowhere equal to $-\infty$, then \bar{x} is an $(\varepsilon + \eta)$ -optimal solution to (PG_{x^*}) , \bar{y}^* is an $(\varepsilon + \eta)$ -optimal solution to (DG_{x^*}) and there is $(\varepsilon + \eta)$ -duality gap for the primal-dual pair of problems $(PG_{x^*}) - (DG_{x^*})$.*
- (b) *If $v \geq 0$, $\bar{x} \in X$ is an ε -optimal solution to (PG_{x^*}) , $\bar{y}^* \in Y^*$ is an η -optimal solution to (DG_{x^*}) and $v(PG_{x^*}) \leq v(DG_{x^*}) + v$, then $(\bar{x}, \bar{y}^*) \in X \times Y^*$ is an $(\eta + \varepsilon + v, \eta + \varepsilon + v)$ -saddle point of $L^{(PG_{x^*})}$.*

PROOF. (a) From Definition 3.1 one gets via Remark 3 that

$$\begin{aligned} \Phi(\bar{x}, 0) - \langle x^*, \bar{x} \rangle - \eta &= \sup_{y^* \in Y^*} L^{(PG_{x^*})}(\bar{x}, y^*) - \eta \leq L^{(PG_{x^*})}(\bar{x}, \bar{y}^*) - \eta \\ (\bar{x}, \bar{y}^*) &\leq \inf_{x \in X} L^{(PG_{x^*})}(x, \bar{y}^*) + \varepsilon = \varepsilon - \Phi^*(x^*, \bar{y}^*). \end{aligned} \quad (12)$$

Using the weak duality for the problems (PG_{x^*}) and (DG_{x^*}) , Equation (12) yields $v(DG_{x^*}) - \eta \leq \varepsilon - \Phi^*(x^*, \bar{y}^*)$ and $\Phi(\bar{x}, 0) - \langle x^*, \bar{x} \rangle - \eta \leq \varepsilon - v(PG_{x^*})$, hence \bar{x} is an $(\varepsilon + \eta)$ -optimal solution to (PG_{x^*}) and \bar{y}^* is an $(\varepsilon + \eta)$ -optimal solution to (DG_{x^*}) . Relation Equation (12) implies also that $\Phi(\bar{x}, 0) - \langle x^*, \bar{x} \rangle - \eta \leq \varepsilon - \Phi^*(x^*, \bar{y}^*)$, consequently $v(PG_{x^*}) \leq v(DG_{x^*}) + \eta + \varepsilon$.

- (b) Using again Remark 3, one obtains that $\Phi(\bar{x}, 0) - \langle x^*, \bar{x} \rangle \geq \sup_{y^* \in Y^*} L^{(PG_{x^*})}(\bar{x}, y^*) \geq L^{(PG_{x^*})}(\bar{x}, \bar{y}^*)$ and $-\Phi^*(0, \bar{y}^*) = \inf_{x \in X} L^{(PG_{x^*})}(x, \bar{y}^*) \leq L^{(PG_{x^*})}(\bar{x}, \bar{y}^*)$. But \bar{x} is an ε -optimal solution to (PG_{x^*}) and \bar{y}^* is an η -optimal solution to (DG_{x^*}) , consequently

$$\begin{aligned} v(DG_{x^*}) - \eta &\leq -\Phi^*(0, \bar{y}^*) \leq L^{(PG_{x^*})}(\bar{x}, \bar{y}^*) \leq \Phi(\bar{x}, 0) \\ &\quad - \langle x^*, \bar{x} \rangle \leq v(PG_{x^*}) + \varepsilon. \end{aligned}$$

Recalling that $v(PG_{x^*}) \leq v(DG_{x^*}) + v$, one obtains from here

$$v(PG_{x^*}) - \eta - v \leq L^{(PG_{x^*})}(\bar{x}, \bar{y}^*) \leq v(DG_{x^*}) + \varepsilon + v,$$

followed by

$$\begin{aligned} \Phi(\bar{x}, 0) - \langle x^*, \bar{x} \rangle - \varepsilon - \eta - v &\leq L^{(PG_{x^*})}(\bar{x}, \bar{y}^*) \\ &\leq -\Phi^*(x^*, \bar{y}^*) + \eta + \varepsilon + v. \end{aligned}$$

Employing again the formulae derived above via Remark 3 one obtains that $(\bar{x}, \bar{y}^*) \in X \times Y^*$ is an $(\eta + \varepsilon + v, \eta + \varepsilon + v)$ -saddle point of $L^{(PG_{x^*})}$. □

If one takes in Theorem 3.17 $\eta = \varepsilon = v = 0$, the two assertions become equivalent, rediscovering [29, Theorem 3.3.2].

Corollary 3.18. *Let $x^* \in X^*$ and assume that Φ is a convex and lower semicontinuous function taking nowhere the value $-\infty$. Then $(\bar{x}, \bar{y}^*) \in X \times Y^*$ is a saddle point of $L^{(PG_{x^*})}$ if and only if \bar{x} is an optimal solution to (PG_{x^*}) , \bar{y}^* is an optimal solution to (DG_{x^*}) and $v(PG_{x^*}) = v(DG_{x^*})$.*

The general scalar optimization problem (PG) encompasses as special cases different classes of scalar optimization problems. We dedicate the next subsections to writing constrained and unconstrained optimization problems as special cases of (PG) and dual problems will be assigned to them by employing carefully chosen perturbation functions (see [19, 29] for more details).

3.1. Constrained Scalar Optimization Problems

Consider the nonempty set $S \subseteq X$ and let the convex cone $C \subseteq Y$ induce a partial ordering on Y . Take the proper functions $f : X \rightarrow \bar{\mathbb{R}}$ and $h : X \rightarrow Y^*$, fulfilling the feasibility condition $\text{dom} f \cap S \cap h^{-1}(-C) \neq \emptyset$. The general primal constrained scalar optimization problem is

$$(PC) \quad \inf_{x \in \mathcal{A}} f(x),$$

whose feasible set is

$$\mathcal{A} = \{x \in S : h(x) \in -C\}.$$

One can find several perturbation functions for which (PC) turns out to be a special case of (PG) . We consider in our presentation two of them, each assigning another dual problem to (PC) that arises from (DG) (cf. [19, 29]).

The classical Lagrange dual problem to (PC) ,

$$(DC^L) \quad \sup_{z^* \in C^*} \inf_{x \in S} [f(x) + (z^*h)(x)],$$

can be seen as a special case of (DG) via the perturbation function

$$\Phi^L : X \times Y \rightarrow \bar{\mathbb{R}}, \quad \Phi^L(x, z) = \begin{cases} f(x), & \text{if } x \in S, h(x) \in z - C, \\ +\infty, & \text{otherwise,} \end{cases}$$

that is proper as f and h are proper and the feasibility condition is satisfied, and whose conjugate is

$$(\Phi^L)^* : X^* \times Y^* \rightarrow \overline{\mathbb{R}}, (\Phi^L)^*(x^*, z^*) = (f - (z^*h) + \delta_S)^*(x^*) + \delta_{C^*}(-z^*).$$

We begin with a characterization via epigraph inclusions of a situation of stable ε -duality gap for the problems (PC) and (DC^L) that is a special case of Theorem 3.1.

Theorem 3.19. *Let W be a subset of X^* . Then it holds*

$$\text{epi}(f + \delta_A)^* \cap (W \times \mathbb{R}) \subseteq \bigcup_{z^* \in C^*} \text{epi}(f + (z^*h))_S^* \cap (W \times \mathbb{R}) - (0, \varepsilon)$$

if and only if for each $x^* \in W$ there exists a $\bar{z}^* \in C^*$ such that

$$(f + \delta_A)^*(x^*) \geq (f + (\bar{z}^*h))_S^*(x^*) - \varepsilon.$$

Remark 11. Analogously, one can particularize the other statements regarding pairs of primal-dual problems $(PG_{x^*}) - (DG_{x^*})$, $x^* \in X^*$, for constrained optimization problems and their Lagrange duals, rediscovering or improving different statements from Boř et al. [20, 39], Boncea and Grad [22], and Jeyakumar and Li [45, 46]. Under additional assumptions which guarantee the convexity of the perturbation function Φ^L (e.g., take S and f convex and h C -convex), the strong duality statement for the problems (PC) and (DC^L) can be derived directly from Corollary 3.11 or Remark 8 by particularizing (RC_i^G) , $i \in \{1, 2, 3, 4\}$ to (cf. [19, 29])

$$(RC_1^L) \mid \exists x' \in \text{dom } f \cap S \text{ such that } h(x') \in -\text{int } C,$$

that is actually the classical Slater constraint qualification,

$$(RC_2^L) \mid \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces, } S \text{ is closed, } f \text{ is lower} \\ \text{semicontinuous, } h \text{ is } C\text{-epi-closed} \\ \text{and } 0 \in \text{sqri}(h(\text{dom } f \cap S \cap \text{dom } h) + C), \end{array}$$

then, when the linear hull of $h(\text{dom } f \cap S \cap \text{dom } h) + C$ is finitely dimensional,

$$(RC_3^L) \mid 0 \in \text{ri}(h(\text{dom } f \cap S \cap \text{dom } h) + C),$$

and

$$(RC_4^L) \mid \begin{array}{l} S \text{ is closed, } f \text{ is lower semicontinuous, } h \text{ is} \\ C\text{-epi-closed and } \bigcup_{z^* \in C^*} \text{epi}(f + (z^*h) + \delta_S)^* \text{ is closed} \\ \text{in the topology } \omega(X^*, X) \times \mathcal{R}. \end{array}$$

Another perturbation function employed to assign a conjugate dual problem to (PC) as a special case of (DG) is (cf. [19, 29])

$$\Phi^{FL} : X \times X \times Y \rightarrow \overline{\mathbb{R}}, \Phi^{FL}(x, y, z) = \begin{cases} f(x + y), & \text{if } x \in S, h(x) \in z - C, \\ +\infty, & \text{otherwise,} \end{cases}$$

that is proper as well because f and h are proper and due to the fulfillment of the mentioned feasibility condition, and has as conjugate the function $(\Phi^{FL})^* : X^* \times X^* \times Y^* \rightarrow \overline{\mathbb{R}}$,

$$(\Phi^{FL})^*(x^*, y^*, z^*) = f^*(y^*) + (-(z^*h) + \delta_S)^*(x^* - y^*) + \delta_{-C^*}(z^*).$$

The dual problem it assigns to (PC) is the Fenchel-Lagrange dual problem

$$(DC^{FL}) \sup_{y^* \in X^*, z^* \in C^*} \{ -f^*(y^*) - ((z^*h) + \delta_S)^*(-y^*) \}.$$

For reader's convenience we give the characterization via epigraph inclusions of a situation of stable ε -duality gap for the problems (PC) and (DC^{FL}) , that is a special case of Theorem 3.1.

Theorem 3.20. *Let W be a subset of X^* . Then it holds*

$$\text{epi}(f + \delta_A)^* \cap (W \times \mathbb{R}) \subseteq \left(\text{epi } f^* + \bigcup_{z^* \in C^*} \text{epi}(z^*h)_S^* \right) \cap (W \times \mathbb{R}) - (0, \varepsilon)$$

if and only if for each $x^* \in W$ there exist $\bar{y}^* \in X^*$ and $\bar{z}^* \in C^*$ such that

$$(f + \delta_A)^*(x^*) \geq f^*(\bar{y}^*) + (\bar{z}^*h)_S^*(x^* - \bar{y}^*) - \varepsilon.$$

Remark 12. Analogously, one can particularize the other statements regarding pairs of primal-dual problems $(PG_{x^*}) - (DG_{x^*})$, $x^* \in X^*$, for constrained optimization problems and their Fenchel-Lagrange duals, rediscovering or improving different statements from Grad [10], Boř et al. [20, 38], and Boncea and Grad [22]. Under additional assumptions which guarantee the convexity of the perturbation function Φ^L (e.g., take S and f convex and h C -convex), the strong duality statement for the problems (PC) and (DC^{FL}) can be derived directly from Corollary 3.11 or Remark 8 by particularizing (RC_i^G) , $i \in \{1, 2, 3, 4\}$ to (cf. [19, 29])

$$(RC_1^{FL}) \mid \exists x' \in \text{dom } f \cap S \text{ such that } f \text{ is continuous at } x' \text{ and } h(x') \in -\text{int } C,$$

$$(RC_2^{FL}) \mid \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces, } S \text{ is closed, } f \text{ is lower} \\ \text{semicontinuous, } h \text{ is } C\text{-epi-closed} \\ \text{and } 0 \in \text{sqri}(\text{dom } f \times C - \text{epi}_{-C}(-h) \cap (S \times Y)), \end{array}$$

then, when $\text{dom } f \times C - \text{epi}_{-C}(-h) \cap (S \times Z)$ is finitely dimensional,

$$(RC_3^{FL}) \mid 0 \in \text{ri}(\text{dom } f \times C - \text{epi}_{-C}(-h) \cap (S \times Z)),$$

and

$$(RC_4^{FL}) \mid \begin{array}{l} S \text{ is closed, } f \text{ is lower semicontinuous, } h \text{ is} \\ C\text{-epi-closed and } \text{epi } f^* + \bigcup_{z^* \in C^*} \text{epi}((z^*h) + \delta_S)^* \text{ is} \\ \text{closed in the topology } \omega(X^*, X) \times \mathcal{R}. \end{array}$$

One can employ other perturbation functions for proposing conjugate dual problems to (PC) as special cases of (DG), too.

For instance, using the perturbation function $\Phi^{EFL} : X \times X \times X \times Y \rightarrow \mathbb{R}$,

$$\Phi^{EFL}(x, y, t, z) = \begin{cases} f(x + y), & \text{if } x \in S, h(x + t) \in z - C, \\ +\infty, & \text{otherwise,} \end{cases}$$

that is proper as well because f and h are proper and due to the fulfillment of the mentioned feasibility condition and has as conjugate the function $(\Phi^{EFL})^* : X^* \times X^* \times X^* \times Y^* \rightarrow \overline{\mathbb{R}}$,

$$(\Phi^{EFL})^*(x^*, y^*, t^*, z^*) = f^*(y^*) + (-z^*h)^*(t^*) + \sigma_S(x^* - y^* - t^*) + \delta_{-C^*}(z^*),$$

one can attach to (PC) is the *extended Fenchel-Lagrange dual problem* (cf. [22, 38])

$$(DC^{EFL}) \quad \sup_{\substack{y^*, t^* \in X^*, \\ z^* \in C^*}} \{ -f^*(y^*) - (z^*h)^*(t^*) - \sigma_S(-y^* - t^*) \},$$

which will not be mentioned further (see [22, 38] for similar statements regarding this dual problem to (PC) that can be rediscovered as special cases of the main ones from this paper).

Remark 13. To give stable ε -duality statements for (PC) and the dual problems we assigned to it within this subsection one can introduce like in Boncea and Grad [22] and Jeyakumar and Li [45, 46] the functions $h^\diamond, h^\diamond_S : X^* \rightarrow \overline{\mathbb{R}}$, defined by $h^\diamond = \inf_{z^* \in C^*} (z^*h)^*$ and $h^\diamond_S = \inf_{z^* \in C^*} (z^*h)_S^*$, respectively. Then, for instance, the stable ε -duality gap for the problems (PC) and (DC^{FL}) is characterized by the epigraph inclusion

$$\text{epi}(f + \delta_A)^* \subseteq \text{epi}(f^* \square h^\diamond_S) - (0, \varepsilon).$$

Remark 14. One can obtain other significant results from the statements presented in this subsection by taking the function f to be identically zero, when characterizations via epigraph inclusions of relations involving the (indicator function of the) feasible set A and, on the other hand, the constraint function h and the constraint set S can be derived, as done in Jeyakumar et al. [8, 9], Grad [10], Boncea and Grad [22], and Boş et al. [38, 39].

3.2. Unconstrained Scalar Optimization Problems

Let the primal *unconstrained* optimization problem

$$(PU) \quad \inf_{x \in X} [f(x) + g(Ax)],$$

where $A : X \rightarrow Y$ is a linear continuous mapping and $f : X \rightarrow \overline{\mathbb{R}}$ and $g : Y \rightarrow \overline{\mathbb{R}}$ are proper functions fulfilling the feasibility condition $\text{dom } f \cap A^{-1}(\text{dom } g) \neq \emptyset$. The perturbation function considered for assigning to (PU) the classical *Fenchel dual problem*

$$(DU) \quad \sup_{y^* \in Y^*} \{ -f^*(A^*y^*) - g^*(-y^*) \},$$

is (cf. [19, 27])

$$\Phi^U : X \times Y \rightarrow \overline{\mathbb{R}}, \quad \Phi^U(x, y) = f(x) + g(Ax + y),$$

which is proper because f and g are proper and due to the fulfillment of the mentioned feasibility condition, and has as conjugate the function

$$(\Phi^U)^* : X^* \times Y^* \rightarrow \overline{\mathbb{R}}, \quad (\Phi^U)^*(x^*, y^*) = f^*(x^* - A^*y^*) + g^*(y^*).$$

Like in the previous subsection, we give only a characterization via epigraph inclusions of a situation of stable ε -duality gap for the problems (PU) and (DU), that is a special case of Theorem 3.1, where the notation $(A^* \times \text{id}_{\mathbb{R}})(\text{epi } g^*) = \{(x^*, r) \in X^* \times \mathbb{R} : \exists y^* \in Y^* \text{ such that } A^*y^* = x^* \text{ and } (y^*, r) \in \text{epi } g^*\}$ is used.

Theorem 3.21. *Let W be a subset of X^* . Then it holds*

$$\text{epi}(f + g \circ A)^* \cap (W \times \mathbb{R}) \subseteq \left(\text{epi } f^* + (A^* \times \text{id}_{\mathbb{R}})(\text{epi } g^*) \right) \cap (W \times \mathbb{R}) - (0, \varepsilon)$$

if and only if for each $x^ \in W$ there exists a $\bar{y}^* \in Y^*$ such that*

$$(f + g \circ A)^*(x^*) \geq f^*(A^*\bar{y}^*) + g^*(\bar{x}^* - \bar{y}^*) - \varepsilon.$$

Remark 15. Analogously, one can particularize the other statements regarding pairs of primal-dual problems $(PG_{x^*}) - (DG_{x^*})$, $x^* \in X^*$, for unconstrained optimization problems and their Fenchel duals, rediscovering or improving different statements from Boş and Wanka [6, 7], Grad [10], Boş et al. [18, 20], and Boncea and Grad [21]. Under additional assumptions which guarantee the convexity of the perturbation function Φ^U (e.g., take f and g convex), the strong duality statement for the problems (PU) and (DU) can be derived directly from Corollary 3.11 or Remark 8 by particularizing (RC_i^G) , $i \in \{1, 2, 3, 4\}$ to (cf. [19, 29])

$$(RC_1^U) \quad \left| \begin{array}{l} \exists x' \in \text{dom } f \cap A^{-1}(\text{dom } g) \text{ such that } g \text{ is} \\ \text{continuous at } Ax', \end{array} \right.$$

$$(RC_2^U) \quad \left| \begin{array}{l} X \text{ and } Y \text{ are Fréchet spaces, } f \text{ and } g \text{ are lower} \\ \text{semicontinuous and } 0 \in \text{sqri}(\text{dom } g - A(\text{dom } f)), \end{array} \right.$$

then, when the linear hull of $\text{dom } g - A(\text{dom } f)$ is finitely dimensional,

$$(RC_3^U) \quad \left| \text{ri } A(\text{dom } f) \cap \text{ri } \text{dom } g \neq \emptyset, \right.$$

condition employed also in the framework of generalized convex optimization, for instance in [47, 48] and

$$(RC_4^U) \quad \left| \begin{array}{l} f \text{ and } g \text{ are lower semicontinuous and } \text{epi } f^* \\ + (A^* \times \text{id}_{\mathbb{R}})(\text{epi } g^*) \\ \text{is closed in the topology } \omega(X^*, X) \times \mathcal{R}. \end{array} \right.$$

Remark 16. Significant particular instances of (PU) can be derived by taking $X = Y$ and A to be the identity mapping on X or f to be identically zero, respectively. The dual problem (DU) and the corresponding duality and optimality conditions statements can be then specialized for these problems, too.

Remark 17. One can alternatively write (PC) as an unconstrained optimization problem

$$(PC) \quad \inf_{x \in X} [f(x) + \delta_{\mathcal{A}}(x)],$$

where the notations are consistent with the ones in the previous subsection. Taking $A := \text{id}_X$, $f := f$ and $g := \delta_{\mathcal{A}}$, a Fenchel dual problem can be attached to (PC) and different duality statements can be obtained for this primal-dual pair of optimization problems as special cases of the ones regarding pairs of primal-dual problems $(PG_{x^*}) - (DG_{x^*})$, $x^* \in X^*$. When necessary, the convexity of the feasible set \mathcal{A} is ensured by taking S convex and h C -convex, while when S is a closed set and h a C -epi-closed vector function, the set \mathcal{A} is closed, too.

Remark 18. The investigations on ε -duality regarding unconstrained problems can be extended for problems consisting in the minimization of a sum of a function with a cone-increasing function composed with a vector function. Considering a convex cone $C \subseteq Y$, a proper function $f : X \rightarrow \overline{\mathbb{R}}$, a proper and C -increasing function $g : Y \rightarrow \overline{\mathbb{R}}$ and a proper vector function $h : X \rightarrow Y^{\bullet}$ fulfilling the feasibility condition $\text{dom } g \cap (h(\text{dom } f) + C) \neq \emptyset$, to the unconstrained composed optimization problem

$$(PO) \quad \inf_{x \in X} [f(x) + g(h(x))],$$

different dual problems that are special cases of (DG) can be attached via perturbation theory. The statements regarding the pairs of primal-dual problems $(PG_{x^*}) - (DG_{x^*})$, $x^* \in X^*$, can be adapted for (PO) and its duals, too, in some instances results from Grad [10], Boţ et al. [18], Boţ [19], and Boncea and Grad [21] being rediscovered. Alternatively, the assertions for (PU) and (DU) can be used for the same purpose by carefully constructing two functions of two variables, say F and G , such that $(F + G)^*(\cdot, 0) = (f + g \circ h)^*$, as done in Boţ et al. [18] and Boţ [19].

4. CHARACTERIZATIONS INVOLVING SUBDIFFERENTIALS

A second large class of closedness type regularity conditions makes use of (ε -)subdifferential inclusions instead of epigraphs. They were initially developed in connection to the notion of total duality for convex optimization problems (see for instance, [8, 38, 39] and some references therein) and in the following we present the most important issues concerning them available up to this moment.

Like in the previous section, we take an $\varepsilon \geq 0$. The first statement we give presents connections between situations of stable ε -duality gap for the problems (PG) and (DG) and ε -subdifferential inclusions. Note that in this case there is no equivalence.

Theorem 4.1. *Let $x \in X$ and $\nu \geq 0$. If*

$$(\Phi(\cdot, 0))^*(x^*) \geq \inf_{y^* \in Y^*} \Phi^*(x^*, y^*) - \varepsilon \quad (13)$$

holds for all $x^ \in \partial_{\nu} \Phi(\cdot, 0)(x)$, one has*

$$\partial_{\nu} \Phi(\cdot, 0)(x) \subseteq \bigcap_{\eta > 0} \text{Pr}_{X^*} \partial_{\varepsilon + \eta + \nu} \Phi(x, 0). \quad (14)$$

Viceversa, Equation (14) yields for any $x^ \in \partial_{\nu} \Phi(\cdot, 0)(x)$ that*

$$(\Phi(\cdot, 0))^*(x^*) \geq \inf_{y^* \in Y^*} \Phi^*(x^*, y^*) - \varepsilon - \nu. \quad (15)$$

PROOF. When $\partial_{\nu} \Phi(\cdot, 0)(x) = \emptyset$ there is nothing to prove.

Assume Equation (13) valid for an $x^* \in \partial_{\nu} \Phi(\cdot, 0)(x)$. This means that for each $\eta > 0$ there exists a $y_{\eta}^* \in Y^*$ such that $(\Phi(\cdot, 0))^*(x^*) \geq \Phi^*(x^*, y_{\eta}^*) - \eta - \nu - \varepsilon$, which, using that $x^* \in \partial_{\nu} \Phi(\cdot, 0)(x)$, yields $\langle x^*, x \rangle + \eta + \varepsilon + \nu \geq \Phi(x, 0) + \Phi^*(x^*, y_{\eta}^*)$, i.e., $(x^*, y_{\eta}^*) \in \partial_{\varepsilon + \eta + \nu} \Phi(x, 0)$. Therefore $x^* \in \bigcap_{\eta > 0} \text{Pr}_{X^*} \partial_{\varepsilon + \eta + \nu} \Phi(x, 0)$, and because x^* was arbitrarily chosen in $\partial_{\nu} \Phi(\cdot, 0)(x)$, (14) follows.

Assume Equation (14) true and let $x^* \in \partial_{\nu} \Phi(\cdot, 0)(x)$. Then for each $\eta > 0$ there exists a $y_{\eta}^* \in Y^*$ such that $(x^*, y_{\eta}^*) \in \partial_{\varepsilon + \eta + \nu} \Phi(x, 0)$. Fixing an $x^* \in \partial_{\nu} \Phi(\cdot, 0)(x)$ and $\eta > 0$, it follows that $\Phi(x, 0) + \Phi^*(x^*, y_{\eta}^*) \leq \langle x^*, x \rangle + \varepsilon + \eta + \nu$, that is equivalent to $\Phi^*(x^*, y_{\eta}^*) - \varepsilon - \nu \leq \langle x^*, x \rangle - \Phi(x, 0) + \eta$, which yields $\inf_{y^* \in Y^*} \Phi^*(x^*, y^*) - \varepsilon - \nu \leq (\Phi(\cdot, 0))^*(x^*) + \eta$. The latter inequality holds for any $\eta > 0$, so letting η tend toward 0 we obtain Equation (15) because x^* was arbitrarily chosen in $\partial_{\nu} \Phi(\cdot, 0)(x)$. \square

In case $\nu = 0$, Theorem 4.1 turns into an equivalence, providing a characterization via subdifferential inclusions of a situation of stable ε -duality gap for the problems (PG) and (DG).

Corollary 4.2. *Let $x \in X$. Then*

$$\partial \Phi(\cdot, 0)(x) \subseteq \bigcap_{\eta > 0} \text{Pr}_{X^*} \partial_{\varepsilon + \eta} \Phi(x, 0) \quad (16)$$

holds if and only if Equation (13) is valid for all $x^ \in \partial \Phi(\cdot, 0)(x)$.*

Remark 19. Let $x \in X$. Employing the weak duality statements for (PG_{x^*}) and (DG_{x^*}) , $x^* \in X^*$, one can show that for any $\nu \geq 0$ it holds $\partial_{\varepsilon + \nu} \Phi(\cdot, 0)(x) \supseteq \bigcap_{\eta > 0} \text{Pr}_{X^*} \partial_{\nu + \varepsilon + \eta} \Phi(x, 0)$. Thus, when $\varepsilon = 0$ Equations (14) and (16) turn into equalities and so does also Equation (13), while Theorem 4.1 yields that if for all $\nu > 0$ one has

$$\partial_{\nu} \Phi(\cdot, 0)(x) = \bigcap_{\eta > 0} \text{Pr}_{X^*} \partial_{\eta + \nu} \Phi(x, 0), \quad (17)$$

then whenever $\mu > 0$ one has $(\Phi(\cdot, 0))^*(x^*) \geq \inf_{y^* \in Y^*} \Phi^*(x^*, y^*) - \mu$ for all $x^* \in \partial_{\mu} \Phi(\cdot, 0)(x)$. Consequently, for all $x^* \in \bigcap_{\mu > 0} \partial_{\mu} \Phi(\cdot, 0)(x) = \partial \Phi(\cdot, 0)(x)$ (here we have used the fact that the intersection regarding $\mu > 0$ of the $\varepsilon + \mu$ -subdifferentials of a function at a point coincides with its ε -subdifferential at that point) it holds $(\Phi(\cdot, 0))^*(x^*) \geq \inf_{y^* \in Y^*} \Phi^*(x^*, y^*)$, which actually turns into an equality since the opposite inequality holds in general and, via Theorem 4.1, yields Equation (17) for all $\nu > 0$. Note also that in Theorem 4.1 and Corollary 4.2 we correct [10, Theorem 2.11 and Theorem 2.12] where Equations (14) and (16) were given as equalities, respectively.

Remark 20. The inequality (Equation 13) can be rewritten as $\nu(PG_{x^*}) \leq \nu(DG_{x^*}) + \varepsilon$, i.e., in Corollary 4.2 an equivalent characterization via subdifferential inclusions of the ε -duality gap for (PG_{x^*}) and (DG_{x^*}) , when $x^* \in \partial\Phi(\cdot, 0)(x)$, i.e., x is an optimal solution to the problem (PG_{x^*}) , is provided.

One can develop Remark 19 even further as follows.

Theorem 4.3. *One has*

$$\partial_\nu\Phi(\cdot, 0)(x) = \bigcap_{\eta>0} \text{Pr}_{X^*} \partial_{\eta+\nu}\Phi(x, 0),$$

for all $x \in X$ and all $\nu > 0$ if and only if for all $x^* \in X^*$ it holds $(\Phi(\cdot, 0))^*(x^*) = \inf_{y^* \in Y^*} \Phi^*(x^*, y^*)$.

PROOF. Let $x \in X$. If $(x, 0) \notin \text{dom } \Phi$, there is nothing to prove, so we consider the case $\Phi(x, 0) \in \mathbb{R}$. Take now $x^* \in X^*$. If $(\Phi(\cdot, 0))^*(x^*) = +\infty$ there is nothing to prove, as $x^* \notin \partial\Phi(\cdot, 0)(x)$, otherwise $x^* \in \partial_\mu\Phi(\cdot, 0)(x)$ for all $\mu \geq \Phi(x, 0) + (\Phi(\cdot, 0))^*(x^*) - \langle x^*, x \rangle$.

The validity of Equation (17) for $\nu = \Phi(x, 0) + (\Phi(\cdot, 0))^*(x^*) - \langle x^*, x \rangle$ implies like in the proof of Theorem 4.1 that $\inf_{y^* \in Y^*} \Phi^*(x^*, y^*) - \nu \leq \langle x^*, x \rangle - \Phi(x, 0) + \eta$ for all $\eta > 0$. Letting η tend toward 0 and replacing ν with its value, it follows $\inf_{y^* \in Y^*} \Phi^*(x^*, y^*) \leq (\Phi(\cdot, 0))^*(x^*)$, which, due to the general validity of the opposite inequality proves the sufficiency.

To show the necessity, let $\nu > 0$ and $x^* \in \partial_\nu\Phi(\cdot, 0)(x)$. Then the hypothesis yields $\Phi(x, 0) + \inf_{y^* \in Y^*} \Phi^*(x^*, y^*) \leq \langle x^*, x \rangle + \nu$. If $\eta > 0$, there exists a $y_\eta^* \in Y^*$ such that $\Phi(x, 0) + \Phi^*(x^*, y_\eta^*) \leq \langle x^*, x \rangle + \nu + \eta$, i.e., $x^* \in \text{Pr}_{X^*} \partial_{\eta+\nu}\Phi(x, 0)$. As η, x and ν were arbitrarily chosen, the conclusion follows via Remark 19. \square

A characterization via subdifferential inclusions of a situation of ε -duality gap for (PG) and (DG) follows.

Theorem 4.4. *Let $x \in X$. Then*

$$\partial_\varepsilon\Phi(\cdot, 0)(x) = \bigcap_{\eta>0} \text{Pr}_{X^*} \partial_{\varepsilon+\eta}\Phi(x, 0)$$

holds if and only if for each $x^* \in \partial_\varepsilon\Phi(\cdot, 0)(x)$ one has

$$\Phi(x, 0) - \langle x^*, x \rangle \leq \sup_{y^* \in Y^*} \{ -\Phi^*(x^*, y^*) \} + \varepsilon. \quad (18)$$

PROOF. Let $x \in X$. If $(x, 0) \notin \text{dom } \Phi$, there is nothing to prove, so we consider the case $\Phi(x, 0) \in \mathbb{R}$.

To show the necessity, let $x^* \in \partial_\varepsilon\Phi(\cdot, 0)(x)$. Then for each $\eta > 0$ there exists a $y_\eta^* \in Y^*$ such that $(x^*, y_\eta^*) \in \partial_{\varepsilon+\eta}\Phi(x, 0)$, i.e., $\Phi(x, 0) + \Phi^*(x^*, y_\eta^*) \leq \langle x^*, x \rangle + \varepsilon + \eta$. This yields $\Phi(x, 0) + \inf_{y^* \in Y^*} \Phi^*(x^*, y^*) \leq \langle x^*, x \rangle + \varepsilon + \eta$ for any $\eta > 0$. Letting η tend toward 0, Equation (18) follows.

To show the opposite implication let $x^* \in \partial_\varepsilon\Phi(\cdot, 0)(x)$. By Equation (18), for each $\eta > 0$ there exists a $y_\eta^* \in Y^*$ such that $\Phi(x, 0) + \Phi^*(x^*, y_\eta^*) \leq \langle x^*, x \rangle + \varepsilon + \eta$, i.e., $(x^*, y_\eta^*) \in \partial_{\varepsilon+\eta}\Phi(x, 0)$. Therefore $x^* \in \bigcap_{\eta>0} \text{Pr}_{X^*} \partial_{\varepsilon+\eta}\Phi(x, 0)$. The reverse inclusion follows by Remark 19. \square

Remark 21. For an $x^* \in \partial_\varepsilon\Phi(\cdot, 0)(x)$, the right-hand side of Equation (18) is actually $\nu(DG_{x^*}) + \varepsilon$, while in the left-hand side we have a quantity that can be larger than or equal to (PG_{x^*}) . Thus, Equation (18) guarantees ε -duality gap for (PG_{x^*}) and (DG_{x^*}) and Theorem 4.4 provides a sufficient condition based on ε -subdifferential inclusions that guarantees it.

Other characterizations via subdifferential inclusions of ε -duality gap situations for (PG) and (DG) follow.

Theorem 4.5. *Let $x \in X$. Then*

$$\partial\Phi(\cdot, 0)(x) \subseteq \text{Pr}_{X^*} \partial_\varepsilon\Phi(x, 0) \quad (19)$$

holds if and only if for each $x^* \in \partial\Phi(\cdot, 0)(x)$ there exists a $y^* \in Y^*$ such that

$$(\Phi(\cdot, 0))^*(x^*) \geq \Phi^*(x^*, y^*) - \varepsilon. \quad (20)$$

PROOF. Inclusion Equation (19) holds if and only if for each $x^* \in \partial\Phi(\cdot, 0)(x)$ there exists a $y^* \in Y^*$ such that $(x^*, y^*) \in \partial_\varepsilon\Phi(x, 0)$, i.e., $\Phi(x, 0) + \Phi^*(x^*, y^*) \leq \langle x^*, x \rangle + \varepsilon$. But $x^* \in \partial\Phi(\cdot, 0)(x)$ if and only if $(\Phi(\cdot, 0))^*(x^*) = \langle x^*, x \rangle - \Phi(x, 0)$ and the desired equivalence follows. \square

Theorem 4.6. *One has*

$$\partial_\nu\Phi(\cdot, 0)(x) = \text{Pr}_{X^*} \partial_\nu\Phi(x, 0) \quad (21)$$

for all $x \in X$ and all $\nu > 0$ if and only if for all $x^* \in X^*$ it holds $(\Phi(\cdot, 0))^*(x^*) = \min_{y^* \in Y^*} \Phi^*(x^*, y^*)$.

PROOF. Let $x \in X$. If $(x, 0) \notin \text{dom } \Phi$, there is nothing to prove, so we consider the case $\Phi(x, 0) \in \mathbb{R}$. Take now $x^* \in X^*$. If $(\Phi(\cdot, 0))^*(x^*) = +\infty$ there is nothing to prove, so we consider further that $(\Phi(\cdot, 0))^*(x^*) \in \mathbb{R}$.

The validity of Equation (21) for $\nu = \Phi(x, 0) + (\Phi(\cdot, 0))^*(x^*) - \langle x^*, x \rangle$ implies the existence of a $\bar{y}^* \in Y^*$ such that $\Phi^*(x^*, \bar{y}^*) \leq \langle x^*, x \rangle - \Phi(x, 0) + \nu$, which actually means $\Phi^*(x^*, \bar{y}^*) \leq \langle x^*, x \rangle - \Phi(x, 0) + \Phi(x, 0) + (\Phi(\cdot, 0))^*(x^*) - \langle x^*, x \rangle$. Consequently, $\Phi^*(x^*, \bar{y}^*) \leq (\Phi(\cdot, 0))^*(x^*)$, which, combined with the always valid inequality $\inf_{y^* \in Y^*} \Phi^*(x^*, y^*) \geq (\Phi(\cdot, 0))^*(x^*)$, yields the sufficiency.

To show the necessity, let $\nu > 0$ and $x^* \in \partial_\nu\Phi(\cdot, 0)(x)$. This yields $\Phi(x, 0) + (\Phi(\cdot, 0))^*(x^*) \leq \langle x^*, x \rangle + \nu$. The hypothesis guarantees the existence of a $\bar{y}^* \in Y^*$ such that $\Phi^*(x^*, \bar{y}^*) = (\Phi(\cdot, 0))^*(x^*)$, consequently $\Phi(x, 0) + \Phi^*(x^*, \bar{y}^*) \leq \langle x^*, x \rangle + \nu$, i.e., $x^* \in \text{Pr}_{X^*} \partial_\nu\Phi(x, 0)$. Since one always has $\partial_\nu\Phi(\cdot, 0)(x) \supseteq \text{Pr}_{X^*} \partial_\nu\Phi(x, 0)$, Equation (21) follows. \square

The following statement can be shown in a similar manner.

Theorem 4.7. *Let $x \in X$. Then*

$$\partial_\varepsilon\Phi(\cdot, 0)(x) = \text{Pr}_{X^*} \partial_\varepsilon\Phi(x, 0)$$

holds if and only if for each $x^* \in \partial_\varepsilon\Phi(\cdot, 0)(x)$ there exists a $\bar{y}^* \in Y^*$ such that

$$\Phi(x, 0) - \langle x^*, x \rangle \leq -\Phi^*(x^*, \bar{y}^*) + \varepsilon.$$

Adding to the function Φ also the classical convexity and topological properties, another characterization of Equation (17) (for an alternative proof by means of epigraph inclusions consult [35, Theorem 3.1]) as well as a consequence of Corollary 4.2 can be delivered.

Theorem 4.8. *Let the function Φ be also convex and lower semicontinuous. The formula (17) is valid for all $x \in X$ and all $v > 0$ if and only if the function $\inf_{y^* \in Y^*} \Phi^*(\cdot, y^*)$ is $\omega(X^*, X)$ -lower semicontinuous.*

PROOF. By Lemma 2.1, the hypotheses yield that the function $(\Phi(\cdot, 0))^*$ is actually the $\omega(X^*, X)$ -lower semicontinuous hull of $\inf_{y^* \in Y^*} \Phi^*(\cdot, y^*)$. The conclusion is then a consequence of Theorem 4.3. \square

Corollary 4.9. *If the function Φ is also convex and lower semicontinuous and the function $\inf_{y^* \in Y^*} \Phi^*(\cdot, y^*)$ is $\omega(X^*, X)$ -lower semicontinuous, then for all $x \in X$ it holds*

$$\partial\Phi(\cdot, 0)(x) = \bigcap_{\eta > 0} \text{Pr}_{X^*} \partial_\eta \Phi(x, 0).$$

Under the same additional assumptions for the function Φ , one can deliver another characterization of Equation (21) that follows via Corollary 3.11 (see also [35]), by means of a closedness type regularity condition this time.

Theorem 4.10. *Let the function Φ be also convex and lower semicontinuous. The formula (21) is valid for all $x \in X$ and all $v > 0$ if and only if the set $\text{Pr}_{X^* \times \mathbb{R}} \text{epi } \Phi^*$ is closed in the topology $\omega(X^*, X) \times \mathcal{R}$.*

Using Theorem 4.5 and Corollary 3.11 one can provide the following statement (see also [35]).

Corollary 4.11. *If Φ is also convex and lower semicontinuous and the set $\text{Pr}_{X^* \times \mathbb{R}} \text{epi } \Phi^*$ is closed in the topology $\omega(X^*, X) \times \mathcal{R}$, then for all $x \in X$ one has*

$$\partial\Phi(\cdot, 0)(x) = \text{Pr}_{X^*} \partial\Phi(x, 0).$$

Remark 22. One can clearer notice the differences between the closedness type regularity conditions considered in Corollary 4.9 and Corollary 4.11 when comparing the way these can be equivalently written as formulae for the conjugate of $\Phi(\cdot, 0)$. The first of them consists of an infimum, thus it characterizes the stable zero duality gap for (PG) and (DG), while the other one means that the same infimum is also attained, i.e., there is stable strong duality for (PG) and (DG), being thus obviously stronger than its counterpart. An example to underline this fact can be found in Boř and Wanka [6]. The difference between these two conditions can be seen also when we equivalently characterize them as formulae for the ε -subdifferential of $\Phi(\cdot, 0)$ in Theorem 4.8 and Theorem 4.10, respectively.

One can employ the results from this subsection for providing ε -optimality conditions for the primal-dual pair (PG) – (DG), too. We begin with a consequence of Theorem 4.1.

Theorem 4.12. (a) *If Equation (3) holds and $\bar{x} \in X$ is an ε -optimal solution to (PG), for each $\eta > 0$ there exists a $\bar{y}_\eta^* \in Y^*$ such that $(0, \bar{y}_\eta^*) \in \partial_{\eta+\varepsilon} \Phi(\bar{x}, 0)$, i.e., $\Phi(\bar{x}, 0) + \Phi^*(0, \bar{y}_\eta^*) \leq \eta + \varepsilon$. Moreover, \bar{y}_η^* is an $\eta + \varepsilon$ -optimal solution to (DG).*
 (b) *If $\bar{x} \in X$ and for each $\eta > 0$ there exists a $\bar{y}_\eta^* \in Y^*$ such that $(0, \bar{y}_\eta^*) \in \partial_{\eta+\varepsilon} \Phi(\bar{x}, 0)$, then \bar{x} is an ε -optimal solution to (PG), each \bar{y}_η^* is an $\eta + \varepsilon$ -optimal solution to (DG) and there is $\eta + \varepsilon$ -duality gap for (PG) and (DG).*

Analogously one can employ Theorem 4.7 in order to achieve ε -optimality conditions for (PG) and (DG), as follows.

Theorem 4.13. (a) *Assuming that the regularity condition (21) is fulfilled and that $\bar{x} \in X$ is an ε -optimal solution to (PG), there exists a $\bar{y}^* \in Y^*$ such that $(0, \bar{y}^*) \in \partial_\varepsilon \Phi(\bar{x}, 0)$, i.e., $\Phi(\bar{x}, 0) + \Phi^*(0, \bar{y}^*) \leq \varepsilon$. Moreover, \bar{y}^* is an ε -optimal solution to (DG).*
 (b) *If $\bar{x} \in X$ and $\bar{y}^* \in Y^*$ fulfill $(0, \bar{y}^*) \in \partial_\varepsilon \Phi(\bar{x}, 0)$, then \bar{x} is an ε -optimal solution to (PG), \bar{y}^* an ε -optimal solution to (DG) and there is ε -duality gap for (PG) and (DG).*

Remark 23. The other statements given in this subsection can be employed for delivering ε -optimality conditions for (PG) and (DG), too. Taking $\varepsilon = 0$ in Theorem 4.13 or in the corresponding statements following from Theorem 4.5, Theorem 4.6 or Theorem 4.10 one rediscovers the optimality condition given in Corollary 3.6.

In the following we particularize the primal problem to be constrained and unconstrained, respectively, as done in the previous section, too.

4.1. Constrained Scalar Optimization Problems

Consider again the framework of Section 3.1. Using first the Lagrange perturbation function Φ^L , one obtains from Theorem 4.4 the following statement where a subdifferential inclusion characterizes a situation of ε -duality gap for (PC) and (DC^L).

Theorem 4.14. *Let $x \in X$. Then*

$$\partial_\varepsilon(f + \delta_{\mathcal{A}})(x) = \bigcap_{\eta > 0} \bigcup_{z^* \in C^*} \partial_{\varepsilon+\eta+(z^*h)(x)}(f + \delta_S + (z^*h))(x)$$

if and only if for each $x^ \in \partial_\varepsilon(f + \delta_{\mathcal{A}})(x)$ one has*

$$f(x) - \langle x^*, x \rangle \leq \sup_{z^* \in C^*} \{ - (f + (\bar{z}^*h))_S^*(x^*) \} + \varepsilon.$$

Analogously one can particularize the other statements involving the (ε) -subdifferential of $\Phi(\cdot, 0)(x)$ to the present framework, too. For instance, Theorem 4.7 turns into the following assertion.

Theorem 4.15. *Let $x \in X$. Then*

$$\partial_\varepsilon(f + \delta_{\mathcal{A}})(x) = \bigcup_{z^* \in C^*} \partial_{\varepsilon+(z^*h)(x)}(f + \delta_S + (z^*h))(x)$$

if and only if for each $x^* \in \partial_\varepsilon(f + \delta_A)(x)$ there exists a $z^* \in C^*$ such that

$$f(x) - \langle x^*, x \rangle \leq -(f + (\bar{z}^*h))_S^*(x^*) + \varepsilon.$$

Adding convexity and topological hypotheses to the functions and sets involved, one obtains the following consequences of Theorem 4.8 and Theorem 4.10, respectively.

Theorem 4.16. *Let S be a closed and convex set, f a convex and lower semicontinuous function and h a C -convex and C -epi-closed vector function. The formula*

$$\partial_\nu(f + \delta_A)(x) = \bigcap_{\eta > 0} \bigcup_{z^* \in C^*} \partial_{\nu+\eta+(z^*h)(x)}(f + \delta_S + (z^*h))(x)$$

is valid for all $x \in X$ and all $\nu > 0$ if and only if the function $\inf_{z^* \in C^*} (f + (\bar{z}^*h))_S^*$ is $\omega(X^*, X)$ -lower semicontinuous.

Theorem 4.17. *Let S be a closed and convex set, f a convex and lower semicontinuous function and h a C -convex and C -epi-closed vector function. The formula*

$$\partial_\nu(f + \delta_A)(x) = \bigcup_{z^* \in C^*} \partial_{\nu+(z^*h)(x)}(f + \delta_S + (z^*h))(x)$$

is valid for all $x \in X$ and all $\nu > 0$ if and only if the set $\cup_{z^* \in C^*} \text{epi}(f + (z^*h) + \delta_S)^*$ is closed in the topology $\omega(X^*, X) \times \mathcal{R}$.

The other perturbation function we employed to assign a conjugate dual problem to (PC) as a special case of (DG) is Φ^{FL} . The statements particularized above for Φ^L become in this case the following ones.

Theorem 4.18. *Let $x \in X$. Then*

$$\partial_\varepsilon(f + \delta_A)(x) = \bigcap_{\eta > 0} \bigcup_{\substack{z^* \in C^*, \varepsilon_1, \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon + \eta + (z^*h)(x)}} (\partial_{\varepsilon_1}f(x) + \partial_{\varepsilon_2}((z^*h) + \delta_S)(x))$$

if and only if for each $x^* \in \partial_\varepsilon(f + \delta_A)(x)$ one has

$$f(x) - \langle x^*, x \rangle \leq \sup_{\substack{z^* \in C^*, \\ y^* \in X^*}} \{ -f^*(y^*) - (\bar{z}^*h)_S^*(x^* - y^*) \} + \varepsilon.$$

Theorem 4.19. *Let $x \in X$. Then*

$$\partial_\varepsilon(f + \delta_A)(x) = \bigcup_{\substack{z^* \in C^*, \varepsilon_1, \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon + \eta + (z^*h)(x)}} (\partial_{\varepsilon_1}f(x) + \partial_{\varepsilon_2}((z^*h) + \delta_S)(x))$$

if and only if for each $x^* \in \partial_\varepsilon(f + \delta_A)(x)$ there exist $z^* \in C^*$ and $y^* \in X^*$ such that

$$f(x) - \langle x^*, x \rangle \leq -f^*(y^*) - (z^*h)_S^*(x^* - y^*) + \varepsilon.$$

Theorem 4.20. *Let S be a closed and convex set, f a convex and lower semicontinuous function and h a C -convex and C -epi-closed vector function. The formula*

$$\partial_\nu(f + \delta_A)(x) = \bigcap_{\eta > 0} \bigcup_{\substack{z^* \in C^*, \varepsilon_1, \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 = \nu + \eta + (z^*h)(x)}} (\partial_{\varepsilon_1}f(x) + \partial_{\varepsilon_2}((z^*h) + \delta_S)(x))$$

is valid for all $x \in X$ and all $\nu > 0$ if and only if the function $\inf_{z^* \in C^*} (f + (\bar{z}^*h))_S^*$ is $\omega(X^*, X)$ -lower semicontinuous.

Theorem 4.21. *Let S be a closed and convex set, f a convex and lower semicontinuous function and h a C -convex and C -epi-closed vector function. The formula*

$$\partial_\nu(f + \delta_A)(x) = \bigcup_{\substack{z^* \in C^*, \varepsilon_1, \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 = \nu + (z^*h)(x)}} (\partial_{\varepsilon_1}f(x) + \partial_{\varepsilon_2}((z^*h) + \delta_S)(x))$$

is valid for all $x \in X$ and all $\nu > 0$ if and only if the set $\text{epi}f^* + \cup_{z^* \in C^*} \text{epi}(z^*h)_S^*$ is closed in the topology $\omega(X^*, X) \times \mathcal{R}$.

Analogously one can particularize the other statements involving the (ε) -subdifferential of $\Phi(\cdot, 0)(x)$ for Φ^{FL} , too.

4.2. Unconstrained Scalar Optimization Problems

Consider now the framework of Section 3.2. From Theorem 4.4 one obtains the following statement where a subdifferential inclusion characterizes a situation of ε -duality gap for (PU) and (DU).

Theorem 4.22. *Let $x \in X$. Then*

$$\partial_\varepsilon(f + g \circ A)(x) = \bigcap_{\eta > 0} \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon + \eta}} (\partial_{\varepsilon_1}f(x) + A^* \partial_{\varepsilon_2}g(Ax))$$

if and only if for each $x^* \in \partial_\varepsilon(f + g \circ A)(x)$ one has

$$f(x) + g(Ax) - \langle x^*, x \rangle \leq \sup_{y^* \in X^*} \{ -f^*(A^*y^*) - g^*(x^* - y^*) \} + \varepsilon.$$

Further, Theorem 4.7 turns into the following assertion.

Theorem 4.23. *Let $x \in X$. Then*

$$\partial_\varepsilon(f + g \circ A)(x) = \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 = \varepsilon}} (\partial_{\varepsilon_1}f(x) + A^* \partial_{\varepsilon_2}g(Ax))$$

if and only if for each $x^* \in \partial_\varepsilon(f + g \circ A)(x)$ there exists a $y^* \in X^*$ such that

$$f(x) + g(Ax) - \langle x^*, x \rangle \leq -f^*(A^*y^*) - g^*(x^* - y^*) + \varepsilon.$$

Adding convexity and topological hypotheses to the functions and sets involved, one obtains the following consequences of Theorem 4.8 and Theorem 4.10, respectively.

Theorem 4.24. *Let the functions f and g be also convex and lower semicontinuous. The formula*

$$\partial_\nu(f + g \circ A)(x) = \bigcap_{\eta > 0} \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 = \nu + \eta}} (\partial_{\varepsilon_1} f(x) + A^* \partial_{\varepsilon_2} g(Ax))$$

is valid for all $x \in X$ and all $\nu > 0$ if and only if the function $\inf_{y^* \in C^*} [f^*(A^*y^*) + g^*(-y^*)]$ is $\omega(X^*, X)$ -lower semicontinuous.

Theorem 4.25. *Let the functions f and g be also convex and lower semicontinuous. The formula*

$$\partial_\nu(f + g \circ A)(x) = \bigcup_{\substack{\varepsilon_1, \varepsilon_2 \geq 0, \\ \varepsilon_1 + \varepsilon_2 = \nu}} (\partial_{\varepsilon_1} f(x) + A^* \partial_{\varepsilon_2} g(Ax))$$

is valid for all $x \in X$ and all $\nu > 0$ if and only if the set $\text{epi} f^* + (A^* \times \text{id}_{\mathbb{R}})(\text{epi} g^*)$ is closed in the topology $\omega(X^*, X) \times \mathcal{R}$.

Analogously one can particularize the other statements involving the (ε) -subdifferential of $\Phi(\cdot, 0)(x)$ to the present framework, too. Moreover, one can see (PC) as an unconstrained optimization problem like in Remark 17 and the corresponding counterparts of the statements given above can be formulated for it, too.

Remark 24. Particularizing the results provided in this section for constrained and unconstrained convex optimization problems (and adding where necessary additional hypotheses) one can rediscover various statements from Jeyakumar et al. [8], Boř et al. [38, 39]. Moreover, the classical *Basic Constraint Qualification* (see for instance, [11–13]) and the *Farkas-Minkowski Constraint Qualification* (cf. [14–16]) prove to be special instances of the regularity conditions presented in this section.

Remark 25. From the statements provided in Sections 3.1 and 3.2 one can derive ε -optimality conditions for (PC) and (PU) and their dual problems, as done in the general case in Theorem 4.12 and Theorem 4.13.

Remark 26. More characterizations of (stable) ε -duality gap and strong/total duality statements via epigraph and/or subdifferential inclusions similar to the ones provided within this chapter for constrained optimization problems can be found in Boncea and Grad [22], while in Boncea and Grad [21] the same kind of assertions are delivered for unconstrained composed optimization problems (see also [49]). Moreover, in Boř and Grad [35] we have provided equivalent characterizations of zero

duality gap and stable strong duality via epigraph inclusions for both constrained and unconstrained, as well for composed optimization problems with the involved functions taken convex.

5. CONCLUSIONS, REMARKS AND FURTHER DIRECTIONS OF RESEARCH

The closedness type regularity conditions have proven during the last decade to be viable alternatives to their more restrictive interiority type counterparts, in both convex optimization and different areas where it was successfully applied. In this survey paper we have deconstructed and reconstructed some closedness type regularity conditions formulated by means of epigraphs and (ε) -subdifferentials, respectively, for general optimization problems, showing thus that they arise naturally when dealing with such problems. Some of the general results were particularized for constrained and unconstrained convex optimization problems, respectively.

Closedness type regularity conditions were employed by different authors in other related research fields, too, like subdifferential calculus (e.g., by [10, 18, 29, 50–52]), DC programming (e.g., by [51, 53–57]), generalized convex optimization (e.g., by [58–60]), semiinfinite programming (e.g., by [15, 16, 50, 53, 61–64]), semidefinite programming (e.g., by [43, 45, 46, 65]), robust optimization (e.g., by [63, 66, 67]), location optimization (e.g., by [68]), vector optimization (e.g., by [10, 29, 63, 64, 69]), monotone operators ([17, 70–78]), machine learning ([79]) or variational inequalities ([80–83]), and the list is far from being complete.

Other possible immediate research fields where we believe that the the closedness type regularity conditions may prove to be useful are bilevel optimization (possibly via the Fenchel-Lagrange approach of [84]), error bounds (maybe via the approach of [85]), equilibrium problems (possibly via variational inequalities, inspired by [80, 82, 83]) and even numerical optimization (e.g., for primal-dual algorithms, by guaranteeing strong duality).

AUTHOR CONTRIBUTIONS

The author S-MG worked alone on this review paper containing also results obtained with/by other authors, that have been properly cited.

FUNDING

Research partially supported by DFG (German Research Foundation), projects WA 922/8-1 and GR3367/4-1.

ACKNOWLEDGMENTS

The author is thankful to two referees for comments and suggestions that have contributed to an improved presentation of the paper.

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